

Oscillations of magnetically levitated aspherical droplets

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In experiments to measure the surface energy of a magnetically levitated molten metal droplet by observation of its oscillation frequencies, Rayleigh's equation is usually used. This assumes that the equilibrium shape is a sphere, and the surface restoring force is due only to surface tension. This work investigates how the vibrations of a non-rotating liquid droplet are affected by the asphericity and additional restoring forces that the levitating field introduces. The calculations show that the expected single frequency of the fundamental mode is split into either three, when there is an axis of rotational symmetry, or five unequally spaced bands. Frequencies, on average, are higher than those of an unconstrained droplet; the surface tension appears to be increased over its normal value. This requires a small correction to be made in all analyses of surface energy. A frequency sum rule is derived from a simplified model of the magnetic field which allows the corresponding Rayleigh frequency to be evaluated from the observed frequencies of the fundamental and translational modes. A more detailed analysis shows a similar correction but one that is also sensitive to the position of the droplet in the field.

1. Introduction

One method of measuring the surface tension of conducting liquids is to levitate a droplet in a magnetic field and measure the frequencies of surface oscillation modes. Use of the Rayleigh equation (Rayleigh 1879), which supposes an unconstrained droplet of spherical geometry, allows the observed frequencies to be linked to the required surface tension.

In a spherical droplet the fundamental mode of oscillation has a five-fold degeneracy. Experimental observations, for example those of Keene, Mills & Brooks (1985), often reveal the presence of three or five closely spaced frequencies and so raise the question of just how reliable or precise are surface tension values derived from the Rayleigh equation.

Levitation is accomplished using an alternating magnetic field with a gradient which allows an equilibrium to be established between the gravitational force on the droplet and the forces generated by the interaction of the induced electric currents with the applied magnetic field. In such conditions several possibilities exist through which oscillation spectra may deviate from those predicted for the unconstrained sphere: the droplet may rotate; the droplet may become aspherical; the surface restoring forces, usually dependent only upon the surface tension, may be modified by the electromagnetic field; or there may be internal motion such as stirring of the material.

This paper deals with the second and third of these possibilities. Rotation of a

spherical droplet is known to split the fundamental into five equally spaced bands (Busse 1984) and is well understood. The problem of stirring is not considered in this work.

Warham (1988) has already made calculations on the oscillations of aspherical droplets of given geometry, and Cheng (1985) has looked at the problem of droplets in an electric field. This paper approaches the deformation problem in a different way from that used by these authors and concentrates on magnetic fields as the origins of asymmetry. Specifically it shows:

- (i) how external forces distort the droplet from sphericity;
- (ii) how frequency spectra are affected by distortions from sphericity;
- (iii) how electromagnetic forces modify the frequency spectra;
- (iv) how to apply the general theory to describe the behaviour of a droplet in a magnetic field of linearly varying intensity.

There are two general approaches to the problem: one is to consider kinetic and potential energies and derive frequencies from the Lagrange equations; the other is to derive an equation of motion from consideration of the forces involved. The second approach is used here and is restricted throughout to small permanent deviations from sphericity, so that linear perturbation theory can be used. Similarly, only small-amplitude oscillations of the deformed droplet are considered. Modifications of the frequency spectrum due to moderate- or large-amplitude nonlinear oscillations of the type reported by such authors as Trinh & Wang (1982) and Tsampopoulos & Brown (1983) are not investigated here.

2. Basic equations

In the calculations which follow, the approach used is first to set up equations which describe how the change in shape of a nearly spherical surface is related to the velocity of the fluid within the bulk of the droplet. The fluid velocity is then related to the pressure inside the droplet by using the hydrodynamic flow equation for a conducting inviscid fluid in an electromagnetic field. This leads to a matrix equation for the surface acceleration in terms of the internal pressure fluctuations. The curvature of the droplet is considered next. This allows both the permanent distortion to the droplet, due to the influence of levitating field, to be found and also yields a second relationship between the motion of the surface and the pressure. The explicit dependence of the surface movements upon the pressure can now be removed and the equation of motion of the surface derived. The final step is to show how the oscillation frequencies can be evaluated from the physical parameters of an actual droplet and its supporting magnetic field.

2.1. Droplet shape and surface acceleration

It will be assumed that the shape of the droplet is only slightly distorted from that of a sphere of radius a , so that in spherical polar coordinates, r, θ, φ , the droplet radius, r_σ , is a function of direction and time, t , given by

$$r_\sigma(\theta, \varphi, t) = a\{1 + R(\theta, \varphi) + \xi(\theta, \varphi, t)\}, \quad (2.1)$$

where $R(\theta, \varphi)$ represents the time-independent deviation from sphericity, due to the effect of the levitating magnetic field, and $\xi(\theta, \varphi, t)$ the time-dependent deviation caused by vibration of the surface. Thus the shape of the droplet is described by the function

$$\sigma(r, \theta, \varphi, t) = r - r_\sigma(\theta, \varphi, t), \quad (2.2)$$

and space is divided into two regions by the surface

$$\sigma(r, \theta, \varphi, t) = 0. \quad (2.3)$$

Points lying outside the droplet conform to the condition $\sigma > 0$, and those inside to $\sigma < 0$.

To set up the equation of motion for the droplet shape consider the total (convective) differential of σ , evaluated at the surface, that is

$$\frac{d\sigma}{dt} = \frac{\partial\sigma}{\partial t} + \mathbf{u} \cdot \nabla\sigma = 0, \quad (2.4)$$

where $\mathbf{u}(x, y, z, t)$ is the bulk fluid velocity. The surface acceleration is therefore

$$-\frac{\partial^2\sigma}{\partial t^2} = \frac{d\mathbf{u}}{dt} \cdot \nabla\sigma \quad (2.5)$$

if time-dependent product terms of order u^2 or higher are neglected, that is, if attention is restricted to small-amplitude oscillations.

2.2. Hydrodynamic flow equation and the magnetic pressure

If the fluid is inviscid, the velocity in the bulk of the droplet is determined by the hydrodynamic flow equation

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p - \rho g \mathbf{k} + \mathbf{J} \wedge \mathbf{B}, \quad (2.6)$$

where p is the pressure, ρ is the density, \mathbf{k} is the unit vector in the upward z -direction, g is the acceleration due to gravity, \mathbf{B} is the magnetic flux density, and \mathbf{J} the electric current density.

Numerical investigation of a magnetically levitated spherical drop by El-Kaddah & Szekely (1983) shows that the current penetrates only a very small region in the drop and the magnetic flux density decreases rapidly below the surface. To model this situation, suppose therefore that the magnetic flux density may be represented as a step function of the form

$$\mathbf{B} = \mathbf{B}_0(r, \theta, \varphi, t) H(\sigma), \quad (2.7)$$

where \mathbf{B}_0 is a vector field yet to be specified and $H(\sigma)$ is the Heaviside step function, so that outside the droplet (where $\sigma > 0$) $\mathbf{B} = \mathbf{B}_0$ and inside the droplet (where $\sigma < 0$) $\mathbf{B} = 0$.

As is shown in the Appendix, a consequence of this assumption is that the electromagnetic forces, now excluded from the bulk of the fluid, make themselves manifest in the equations as an effective magnetic pressure, $p^\dagger(\theta, \varphi, t)$, on the surface of the droplet with magnitude

$$p^\dagger(\varphi, \theta, t) = B^2(\text{surface})/2\mu_0 \quad (2.8)$$

where μ_0 is the permittivity of free space. Experimentally, Colgate, Furth & Halliday (1960) have found good evidence for just such a pressure.

This magnetic pressure acts in addition to that generated inside the curved surface of the droplet by the surface tension, γ , giving for the total pressure just below the surface of the droplet the boundary condition

$$p(\text{surface}) = \gamma \nabla \cdot \hat{\mathbf{n}} + p^\dagger. \quad (2.9)$$

Here $\hat{\mathbf{n}}$ is the outwardly directed normal to the surface and $\nabla \cdot \hat{\mathbf{n}}$ its curvature.

Since the electromagnetic forces, $\mathbf{J} \wedge \mathbf{B}$, no longer directly appear in the hydrodynamic flow equation, splitting the pressure into a time-independent part, $p_s(r, \theta, \varphi)$, and a time-dependent part, $p_t(r, \theta, \varphi, t)$, gives

$$\frac{d\mathbf{u}}{dt} = -\frac{1}{\rho} \nabla(p_s + p_t) - g\mathbf{k}, \quad (2.10)$$

which yields from the time-independent terms the hydrostatic pressure

$$p_s = -\rho g z + p_0, \quad (2.11)$$

where p_0 is a constant, and from the time-dependent ones

$$\frac{d\mathbf{u}}{dt} = -\frac{1}{\rho} \nabla p_t. \quad (2.12)$$

For an incompressible fluid $\nabla \cdot \mathbf{u} = 0$, so this last equation implies that $\nabla^2 p_t = 0$. Consequently the time-dependent part of the pressure can be expressed as the series

$$p_t = \sum_{lm} \left(\frac{r}{a}\right)^l Y_l^m(\theta, \varphi) \alpha_l^m(t), \quad (2.13)$$

where $Y_l^m(\theta, \varphi)$ are complex spherical harmonics, and $\alpha_l^m(t)$ are time-dependent functions.

2.3. Matrix description

The above expression can be used to generate a matrix equation for the surface acceleration. From (2.5) and (2.12) the surface acceleration is

$$\frac{\partial^2 \sigma}{\partial t^2} = \frac{1}{\rho} \nabla p_t \cdot \nabla \sigma. \quad (2.14)$$

The two gradient vectors, ∇p and $\nabla \sigma$, can be written in terms of vectors parallel and perpendicular to the position vector, $\mathbf{r} = r\hat{\mathbf{r}}$. This gives

$$\nabla p = \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \nabla p) - \hat{\mathbf{r}} \wedge (\hat{\mathbf{r}} \wedge \nabla p) \quad (2.15)$$

$$= \hat{\mathbf{r}} \frac{\partial p}{\partial r} - \frac{i}{r} \hat{\mathbf{r}} \wedge \hat{\mathbf{L}} p, \quad (2.16)$$

where the operator $\hat{\mathbf{L}}$ is defined by

$$\hat{\mathbf{L}} = -i(\mathbf{r} \wedge \nabla) = i \left(\frac{\hat{\boldsymbol{\theta}}}{\sin \theta} \frac{\partial}{\partial \varphi} - \hat{\boldsymbol{\phi}} \frac{\partial}{\partial \theta} \right). \quad (2.17)$$

Similarly:

$$\nabla \sigma = \hat{\mathbf{r}} - \frac{i}{r} \hat{\mathbf{r}} \wedge \hat{\mathbf{L}} \sigma. \quad (2.18)$$

Equation (2.14) may thus be written

$$\frac{\partial^2 \sigma}{\partial t^2} = \frac{\partial p_t}{\partial r} - \frac{1}{r^2} (\hat{\mathbf{L}} p_t) \cdot (\hat{\mathbf{L}} \sigma), \quad (2.19)$$

and when expressed in terms of ζ and the $\alpha_l^m(t)$ becomes

$$-\rho a \frac{\partial^2 \zeta}{\partial t^2} = \sum_l \frac{l}{a} \left(\frac{r}{a}\right)^{l-1} Y_l^m \alpha_l^m + \frac{a}{r^2} \sum_l \left(\frac{r}{a}\right)^l \hat{\mathbf{L}}(R + \zeta) \cdot \hat{\mathbf{L}} Y_l^m \alpha_l^m. \quad (2.20)$$

This equations refers to the surface, so $r = r_\sigma = a(1 + R + \zeta)$. Terms of order higher than R^2 may be dropped, as may all products of time-dependent terms such as $\zeta(t)\alpha_l^m(t)$ and ζ^2 . This leaves

$$-\rho a \frac{\partial^2 \zeta}{\partial t^2} = \sum \frac{l}{a} [1 + (l-1)R] Y_l^m \alpha_l^m + \frac{1}{a} \sum (\hat{\mathbf{L}}R) \cdot \hat{\mathbf{L}} Y_l^m \alpha_l^m. \tag{2.21}$$

With the usual notation for the integral:

$$\int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta (Y_j^k)^* f(\theta, \varphi) Y_l^m = \langle j, k | f(\theta, \varphi) | l, m \rangle, \tag{2.22}$$

left multiplying (2.21) by $(Y_j^k)^*$ and integrating gives

$$-\rho a^2 \langle j, k | \partial^2 \zeta / \partial t^2 \rangle = \sum \{ l \langle j, k | l, m \rangle + l(l-1) \langle j, k | R | l, m \rangle + \langle j, k | (\hat{\mathbf{L}}R) \cdot \hat{\mathbf{L}} | l, m \rangle \} \alpha_l^m. \tag{2.23}$$

This is a matrix equation of the form

$$-\rho a^2 \frac{\partial^2}{\partial t^2} \begin{pmatrix} \vdots \\ \langle j, k | \zeta \rangle \\ \vdots \end{pmatrix} = \mathcal{A} \begin{pmatrix} \vdots \\ \alpha_l^m \\ \vdots \end{pmatrix} \tag{2.24}$$

with elements

$$\mathcal{A}_{jk,lm} = l \langle j, k | l, m \rangle + l(l-1) \langle j, k | R | l, m \rangle + \langle j, k | (\hat{\mathbf{L}}R) \cdot \hat{\mathbf{L}} | l, m \rangle. \tag{2.25}$$

2.4 Curvature and droplet distortion

To complete the matrix equation of motion for the surface, the α_l^m terms must be eliminated from the above equation. To do this consider the boundary condition (2.9) and express the curvature, $\nabla \cdot \hat{\mathbf{n}}$, in terms of ζ . Since

$$\hat{\mathbf{n}} = \nabla \sigma / \|\nabla \sigma\| \tag{2.26}$$

and, from (2.18) $\|\nabla \sigma\| = [1 - (1/r^2)(\hat{\mathbf{L}}\sigma)^2]^{\frac{1}{2}}$ (2.27)

it follows that, provided the total deviation from sphericity remains small,

$$\hat{\mathbf{n}} = \nabla \sigma \{ 1 + (1/2r^2)(\hat{\mathbf{L}}\sigma)^2 \} \tag{2.28}$$

and hence $\nabla \cdot \hat{\mathbf{n}} = \nabla^2 \sigma + \frac{1}{2} \left\{ \nabla^2 \sigma \left(\frac{\hat{\mathbf{L}}\sigma}{r} \right)^2 + \nabla \sigma \cdot \nabla \left(\frac{\hat{\mathbf{L}}\sigma}{r} \right)^2 \right\}.$ (2.29)

Since $\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) - \frac{1}{r^2} \hat{\mathbf{L}}^2,$ (2.30)

where $\hat{\mathbf{L}}^2 = \hat{\mathbf{L}} \cdot \hat{\mathbf{L}} = - \left(\frac{1}{\sin \theta} \right) \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) - \left(\frac{1}{\sin^2 \theta} \right) \frac{\partial^2}{\partial \varphi^2},$ (2.31)

the curvature is therefore

$$\nabla \cdot \hat{\mathbf{n}} = \nabla^2 \sigma - \frac{1}{2r^4} \{ \hat{\mathbf{L}}^2 \sigma (\hat{\mathbf{L}}\sigma)^2 + \hat{\mathbf{L}}\sigma \cdot \hat{\mathbf{L}} (\hat{\mathbf{L}}\sigma)^2 \}. \tag{2.32}$$

Here the second group of terms are of order $(R + \zeta)^3$ and can be neglected, so

$$\nabla \cdot \hat{\mathbf{n}} = \nabla^2 \sigma = \frac{2}{r} + \frac{a}{r^2} \hat{\mathbf{L}}^2 (R + \zeta). \tag{2.33}$$

At the surface, $r = r_\sigma$, so using (2.1) this gives

$$\nabla \cdot \hat{\mathbf{n}} = \frac{2}{a} (1 - (\zeta + R) + (\zeta + R)^2 - \dots) + \frac{1}{a} (1 - 2(\zeta + R) + \dots) \hat{\mathbf{L}}^2 (R + \zeta) \tag{2.34}$$

or
$$a\nabla \cdot \hat{n} = 2 + (\hat{L}^2 - 2)R + (\hat{L}^2 - 2)\zeta + 4\zeta R - 2\zeta\hat{L}^2 R - 2R\hat{L}^2 \zeta. \quad (2.35)$$

This equation describes how the curvature depends on the deviation from sphericity.

2.5. Magnetic pressure and permanent distortions

It is also necessary to describe how the magnetic pressure changes as the droplet deforms with time. Since ζ can be expanded as

$$\zeta = \sum Y_l^m \langle l, m | \zeta \rangle, \quad (2.36)$$

p^\dagger can be split into a time-independent part, p_s^\dagger , and a time-dependent part of the general form

$$p^\dagger(\theta, \varphi, t) = p_s^\dagger + \sum_{lm} \langle l, m | \zeta \rangle \left(\frac{\partial p^\dagger}{\partial \langle l, m | \zeta \rangle} \right)_{\zeta=0}. \quad (2.37)$$

Bringing together the results of (2.11), (2.13) and (2.35), the boundary condition (2.9) becomes

$$p_0 - \rho g z + \sum_{lm} (l + lR) Y_l^m \alpha_l^m = p_s^\dagger + \sum_{lm} \frac{\partial p^\dagger}{\partial \langle l, m | \zeta \rangle} \langle l, m | \zeta \rangle + \left(\frac{\gamma}{a} \right) \{ 2 + (\hat{L}^2 - 2)R + (\hat{L}^2 - 2)\zeta - 2R(\hat{L}^2 - 2)\zeta - 2(\hat{L}^2 R)\zeta \}. \quad (2.38)$$

This result can be used to find the permanent distortion, R , of the droplet. At the surface

$$z = r_\sigma \cos \theta = a(1 + R + \zeta) \left(\frac{4\pi}{3} \right)^{\frac{1}{2}} Y_1^0, \quad (2.39)$$

the permanent deformation can be expanded by writing the $R(\theta, \varphi)$ in terms of spherical harmonics:

$$R(\theta, \varphi) = \sum \langle d, e | R \rangle Y_d^e, \quad (2.40)$$

and since

$$\hat{L}^2 Y_d^e = d(d+1) Y_d^e \quad (2.41)$$

consideration of the time-independent terms in (2.38) requires that

$$p_0 - \rho g a(1 + R) \left(\frac{4\pi}{3} \right)^{\frac{1}{2}} Y_1^0 = \frac{\gamma}{a} \left\{ 2 + \sum_{de} Y_d^e (d-1)(d+2) \langle d, e | R \rangle \right\} + p_s^\dagger; \quad (2.42)$$

hence for $d > 1$, and correct to first order in R

$$\langle d, e | R \rangle = - \left(\frac{a}{\gamma} \right) \frac{\langle d, e | p_s^\dagger \rangle}{(d-1)(d+2)}. \quad (2.43)$$

Also from (2.42) it can be seen that for equilibrium, the net lifting force, F_z , which balances the weight of the droplet, requires

$$F_z = \frac{4}{3}\pi a^3 \rho g = - \left(\frac{4}{3}\pi \right)^{\frac{1}{2}} a^2 \langle 1, 0 | p_s^\dagger \rangle \{ 1 - \langle 1, 0 | R | 1, 0 \rangle \}. \quad (2.44)$$

2.6 Matrix equation of motion

The matrix equation of motion can now be completed. Consideration of the time-dependent terms in (2.38) shows that there is a matrix equation of the form

$$\mathcal{B} \begin{pmatrix} \vdots \\ \alpha_l^m \\ \vdots \end{pmatrix} = \mathcal{C} \begin{pmatrix} \vdots \\ \langle l, m | \zeta \rangle \\ \vdots \end{pmatrix} \begin{pmatrix} \gamma \\ a \end{pmatrix}, \quad (2.45)$$

where the matrix elements of \mathcal{B} are

$$\mathcal{B}_{jk,lm} = \langle j, k | l, m \rangle + l \langle j, k | R | l, m \rangle \quad (2.46)$$

and those of \mathcal{C} are

$$\begin{aligned} \mathcal{C}_{jk,lm} = & (l-1)(l+2) \{ \langle j, k | l, m \rangle - 2 \langle j, k | R | l, m \rangle \} - 2 \langle j, k | (\hat{\mathbf{L}}^2 R) | l, m \rangle \\ & + (a/\gamma) \frac{\partial \langle j, k | p^\dagger \rangle}{\partial \langle l, m | \zeta \rangle} + \frac{\rho g a^2}{\gamma} \left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \langle j, k | 1, 0 | l, m \rangle. \end{aligned} \quad (2.47)$$

The α_l^m terms can therefore be substituted into (2.24) to give a matrix differential equation for the components of ζ of the form

$$-\frac{\partial^2}{\partial t^2} \begin{pmatrix} \vdots \\ \langle j, k | \zeta \rangle \\ \vdots \end{pmatrix} = \mathcal{A} \mathcal{B}^{-1} \mathcal{C} \begin{pmatrix} \vdots \\ \langle j, k | \zeta \rangle \\ \vdots \end{pmatrix} \left(\frac{\gamma}{\rho a^3} \right). \quad (2.48)$$

This is the required equation of motion. It is the equation of a set of harmonic oscillators, such that if λ_q is the q th eigenvalue of $\mathcal{A} \mathcal{B}^{-1} \mathcal{C}$, the oscillation frequency, ω_q is given by

$$\omega_q^2 = \lambda_q (\gamma / \rho a^3). \quad (2.49)$$

The corresponding eigenvector, ζ_q , gives the shape of the q th normal mode of vibration:

$$\zeta_q(\theta, \varphi) = \sum Y_j^k \langle j, k | \zeta_q \rangle, \quad (2.50)$$

and so the total time-dependent deviation from sphericity, ζ , is

$$\zeta(\theta, \varphi, t) = \sum_q A_q \zeta_q(\theta, \varphi) \exp(i\omega_q t), \quad (2.51)$$

where A_q is the amplitude of the q th normal mode. This result describes how the surface of the droplet behaves with time; it is a linear combination of these normal modes of vibration.

2.7. The normal modes

In order to determine the form and oscillation frequency of each normal mode it is necessary to find the eigenvectors and eigenvalues of the matrix $\mathcal{A} \mathcal{B}^{-1} \mathcal{C}$. To do this, first invert the matrix \mathcal{B} to give, correct to first order in R ,

$$(\mathcal{B}^{-1})_{jk,lm} = \langle j, k | l, m \rangle - l \langle j, k | R | l, m \rangle, \quad (2.52)$$

then, multiplying out,

$$(\mathcal{A} \mathcal{B}^{-1})_{jk,lm} = l \langle j, k | l, m \rangle + (l^2 - l - jl) \langle j, k | R | l, m \rangle + \langle j, k | (\hat{\mathbf{L}} R) \cdot \hat{\mathbf{L}} | l, m \rangle, \quad (2.53)$$

and hence

$$\begin{aligned} (\mathcal{A} \mathcal{B}^{-1} \mathcal{C})_{jk,lm} = & l(l-1)(l+2) \langle j, k | l, m \rangle + (l-1)(l+2) \langle j, k | (\hat{\mathbf{L}} R) \cdot \hat{\mathbf{L}} | l, m \rangle \\ & + (l-1)(l+2)(l^2 - l - jl - 2j) \langle j, k | R | l, m \rangle - 2j \langle j, k | (\hat{\mathbf{L}}^2 R) | l, m \rangle \\ & + l \left(\frac{a}{\gamma} \right) \frac{\partial \langle j, k | p^\dagger \rangle}{\partial \langle l, m | \zeta \rangle} + l \frac{\rho g a^2}{\gamma} \left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \langle j, k | 1, 0 | l, m \rangle. \end{aligned} \quad (2.54)$$

The term $\langle j, k | (\hat{\mathbf{L}} R) \cdot \hat{\mathbf{L}} | l, m \rangle$ is evaluated by use of the identity

$$\hat{\mathbf{L}}^2(fg) = \hat{\mathbf{L}} \cdot (\hat{\mathbf{L}}(fg)) = f(\hat{\mathbf{L}}^2 g) + g(\hat{\mathbf{L}}^2 f) + 2(\hat{\mathbf{L}} f) \cdot (\hat{\mathbf{L}} g), \quad (2.55)$$

which gives

$$2\langle j, k | (\hat{\mathbf{L}}R) \cdot \hat{\mathbf{L}} | l, m \rangle = j(j+1)\langle j, k | R | l, m \rangle - l(l+1)\langle j, k | R | l, m \rangle - \langle j, k | (\hat{\mathbf{L}}^2 R) | l, m \rangle. \tag{2.56}$$

For the purpose of finding the eigenvalues, the results of first-order perturbation theory are sufficient, so all off-diagonal terms between the non-degenerate unperturbed eigenvectors can be ignored. This corresponds to the condition $j = l$, and hence the block-diagonal submatrices of $\mathcal{A}\mathcal{B}^{-1}\mathcal{C}$ are

$$(\mathcal{A}\mathcal{B}^{-1}\mathcal{C})_{lk,lm} = l(l-1)(l+2)\delta_{km} - 3l(l-1)(l+2)\langle l, k | R | l, m \rangle - \frac{1}{2}(l^2 + 5l - 2)\langle l, k | (\hat{\mathbf{L}}^2 R) | l, m \rangle + l \left(\frac{a}{\gamma} \right) \left(\frac{\partial \langle l, k | p^\dagger \rangle}{\partial \langle l, m | \zeta \rangle} \right)_{\zeta=0}, \tag{2.57}$$

where δ_{km} is the Kronecker delta function. Note that all gravity terms have disappeared from the equation since integrals of the type $\langle l, k | 1, 0 | l, m \rangle$ are equal to zero.

This result can be used to evaluate the form of the normal modes and associated oscillation frequencies of an actual droplet provided that the permanent distortion from sphericity, R , is known, and information is also available about how the magnetic pressure varies with any change in shape of the droplet.

3. Simple applications of the theory

The parameters required above are not commonly available. It should be possible by direct observation of a levitated droplet to measure its permanent distortion, but there remains unresolved the problem of the magnetic pressure terms. This prompts the question as to whether it is possible at this point in the analysis to obtain some useful results by making some simplifying assumption about the behaviour of the magnetic pressure. The following sections explore two such simplifications before moving on to a complete description of the magnetic pressure. They are

- (i) to neglect those terms which involve differentials of the magnetic pressure in (2.57); physically this is equivalent to the magnetic pressure at the surface of the droplet remaining constant as the droplet oscillates;
- (ii) to assume that both the induced component of the magnetic field and the original supporting magnetic field remain unchanged as the droplet oscillates.

3.1. Constant magnetic pressure: the $l = 2$ fundamental vibrations

To obtain a first approximation to the oscillation frequencies, suppose that the magnetic pressure at the surface of the droplet is constant. The number of parameters affecting the frequency spectrum is considerably reduced. With this approximation, $\partial p^\dagger / \partial \langle l, m | \zeta \rangle = 0$, and from (2.57)

$$(\mathcal{A}\mathcal{B}^{-1}\mathcal{C})_{lk,lm} = l(l-1)(l+2)\delta_{km} - \sum \langle d, e | R \rangle \langle l, k | d, e | l, m \rangle \{ 3l(l-1)(l+2) + \frac{1}{2}d(d+1)(l^2 + 5l - 2) \}. \tag{3.1}$$

Two consistency checks can be applied to this equation.

(i) Rayleigh's result can be recovered by setting $R = 0$. $\mathcal{A}\mathcal{B}^{-1}\mathcal{C}$ is now a diagonal matrix, with eigenvalues $\lambda_l = l(l-1)(l+2)$, and so from (2.49) the frequencies, as expected, are

$$\omega_l^2 = l(l-1)(l+2)(\gamma/\rho a^3). \tag{3.2}$$

(ii) An additional check on the calculation can be made by setting $R = \delta a/a$,

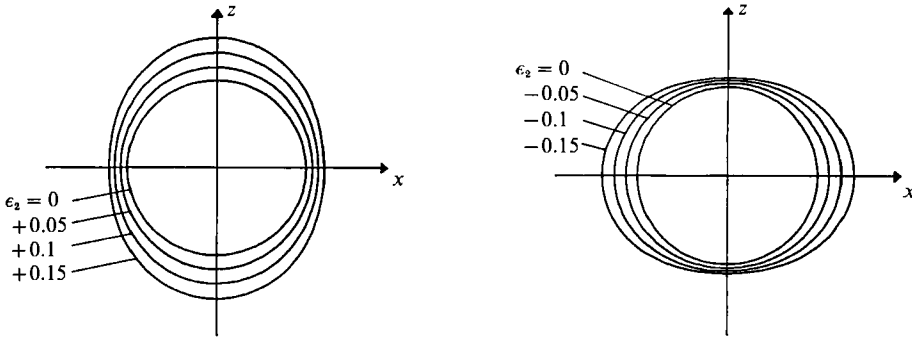


FIGURE 1. Sections in the (x, z) -plane of a permanently distorted droplet with radius at the surface given by $r_s = a(1 + \epsilon_2 Y_2^0)$ for various values of ϵ_2 . (For clarity of presentation, a has been successively increased by 10% but the actual distortion conserves the volume of the droplet.)

that is, to a small increase in radius of the spherical droplet. With this distortion, $\langle j, k | R | l, m \rangle = (\delta a/a) \delta_{jl} \delta_{km}$, and from (3.1)

$$\omega_l^2 = l(l-1)(l+2) \left(1 - 3 \frac{\delta a}{a} \right) (\gamma/\rho a^3). \tag{3.3}$$

This is the result that would be obtained by differentiating Rayleigh's equation directly, (3.2).

In the sequence of oscillation modes, the condition $l = 0$ is forbidden since it describes a change of radius denied by the condition that the droplet is of constant density. The next mode, when $l = 1$ describes translational motion of the centre of mass, and does not correspond to an oscillation of the surface of the droplet itself. Among superficial vibrations the fundamental therefore corresponds to the condition $l = 2$, and for these modes

$$(\mathcal{A} \mathcal{B}^{-1} \mathcal{C})_{2k, 2m} = 8 \delta_{km} - \sum_{de} \langle d, e | R \rangle \langle 2, k | d, e | 2, m \rangle \{24 + 6d(d+1)\}. \tag{3.4}$$

These particular integrals are only non-zero for $d = 0, 2, 4$ in the above equation and since the condition $d = 0$ corresponds to a change in radius, so only distortions from sphericity which contain terms with $d = 2$ and 4 need be considered.

The number of arbitrary parameters can be further reduced if the z -axis is preserved as a rotational axis of symmetry. This condition corresponds to $e = 0$, and hence the only deformation of cylindrical symmetry that can affect the fundamental modes is of the form

$$R = \epsilon_2 Y_2^0 + \epsilon_4 Y_4^0, \tag{3.5}$$

where $\epsilon_2 = \langle 2, 0 | R \rangle$ and $\epsilon_4 = \langle 4, 0 | R \rangle$ are two as yet undetermined constants. The droplet shapes corresponding to these two types of permanent distortion are shown in figures 1 and 2 as sections of the (x, z) -plane.

Under the condition of $e = 0$ there can be no off-diagonal terms: the only non-zero elements occur when $k = m$ (values of the required integrals are given in table 2 in the Appendix). Hence the eigenvalues, λ_q , are simply the diagonal elements $(\mathcal{A} \mathcal{B}^{-1} \mathcal{C})_{2m, 2m}$:

$$\left. \begin{aligned} (\mathcal{A} \mathcal{B}^{-1} \mathcal{C})_{20, 20} &= 8 - (5/\pi)^{1/2} (60\epsilon_2/7) - 432\epsilon_4/7\pi^{1/2}, \\ (\mathcal{A} \mathcal{B}^{-1} \mathcal{C})_{2\pm 1, 2\pm 1} &= 8 - (5/\pi)^{1/2} (30\epsilon_2/7) + 288\epsilon_4/7\pi^{1/2}, \\ (\mathcal{A} \mathcal{B}^{-1} \mathcal{C})_{2\pm 2, 2\pm 2} &= 8 + (5/\pi)^{1/2} (60\epsilon_2/7) - 72\epsilon_4/7\pi^{1/2}. \end{aligned} \right\} \tag{3.6}$$

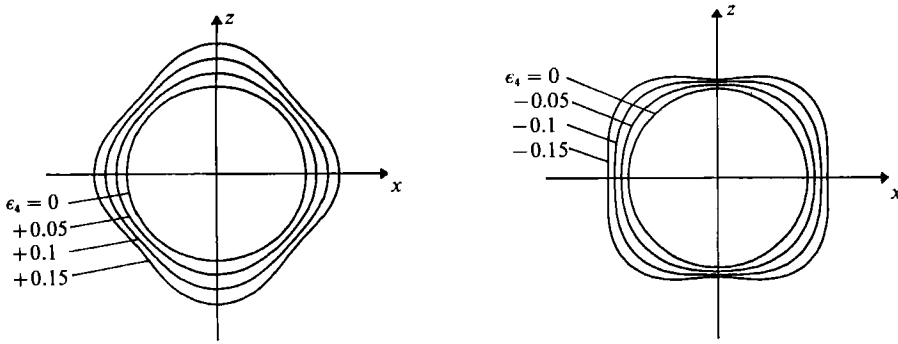


FIGURE 2. Sections in the (x, z) -plane of a permanently distorted droplet with radius at the surface given by $r_\sigma = a(1 + \epsilon_4 Y_4^0)$ for various values of ϵ_4 . (For clarity of presentation, a has been successively increased by 10% but the actual distortion conserves the volume of the droplet.)

m	Frequency	Symmetry	Normal mode	Intensity ($\theta = \frac{1}{2}\pi$)
0	ω_Σ	Σ^+	$d_z^2 = (5/16\pi)^{\frac{1}{2}}(3 \cos^2 \theta - 1)$	High
± 1	ω_Π	Π	$\begin{cases} d_{xz} = (15/4\pi)^{\frac{1}{2}} \sin \theta \cos \theta \cos \varphi \\ d_{yz} = (15/4\pi)^{\frac{1}{2}} \sin \theta \cos \theta \sin \varphi \end{cases}$	Low
± 2	ω_Δ	Δ	$\begin{cases} d_{x^2-y^2} = (15/16\pi)^{\frac{1}{2}} \sin^2 \theta \cos 2\varphi \\ d_{xy} = (15/16\pi)^{\frac{1}{2}} \sin^2 \theta \sin 2\varphi \end{cases}$	High

TABLE 1. Droplet possessing cylindrical symmetry

The forms of the five corresponding normal modes are shown in table 1. They are the same as the normalized angular wave functions of the atomic d orbitals. Owing to the presence of the axis of rotational symmetry in the droplet, they form a basis for a representation of the group $C_{\infty v}$ which contains the irreducible representations Σ^+ , Π , and Δ (e.g. Cotton 1963). Because of the double degeneracy of the Π and Δ symmetry species, there are only three different fundamental frequencies. For convenience, these have been designated ω_Σ , ω_Π and ω_Δ , and correspond to $m = 0$, $m = \pm 1$ and $m = \pm 2$ respectively.

From (2.49) these angular frequencies are

$$\left. \begin{aligned} \omega_\Sigma &= \omega_R(1 - 0.6758\epsilon_2 - 2.176\epsilon_4), \\ \omega_\Pi &= \omega_R(1 - 0.3379\epsilon_2 + 1.4507\epsilon_4), \\ \omega_\Delta &= \omega_R(1 + 0.6758\epsilon_2 - 0.3627\epsilon_4), \end{aligned} \right\} \quad (3.7)$$

where ω_R is the angular frequency of the $l = 2$ mode of the undistorted sphere, the Rayleigh frequency, and is given by

$$\omega_R = 2\pi\nu_R = (8\gamma/\rho a^3)^{\frac{1}{2}}. \quad (3.8)$$

These results are in agreement with the work of Warham, who used the minimization of surface energy approach to the calculations. They provide a good check on the general technique.

An estimate of the relative intensity of each mode can be made for any particular axis of observation. For observations down the z -axis for example, measurements will be most sensitive to changes in the diameter of the droplet in the (x, y) -plane, i.e. $\theta = \frac{1}{2}\pi$. For this case the relative intensities expected are shown in table 1.

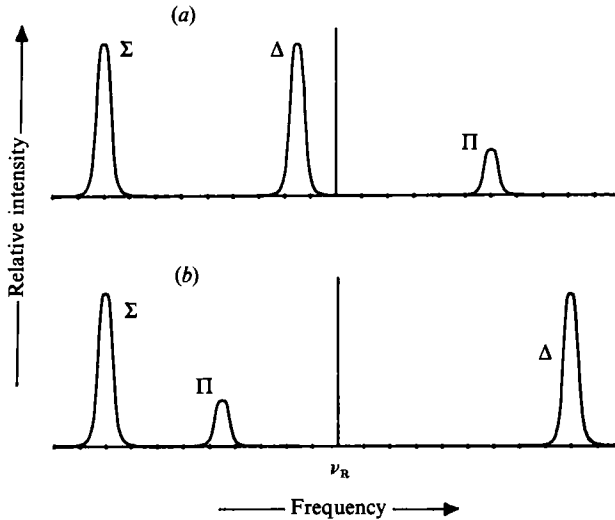


FIGURE 3. Spectra expected for particular values of the ϵ_2 and ϵ_4 terms when observations are made from above. (a) $\epsilon_2 = 0$, $\epsilon_4 > 0$, symmetrically split spectrum. (b) $\epsilon_2 > 0$, $\epsilon_4 = 0$, asymmetrically split spectrum.

Some points to note are that: (i) as a consequence of the rotational axis of symmetry a maximum of three fundamental vibration bands are predicted to be seen; (ii) there is no *a priori* reason for the spectrum to be symmetrically split, in the sense of having equally spaced bands; (iii) when the distortion has only a Y_2^0 component, the spacing between the bands is in the ratio 3:1. When the distortion consists only of a Y_4^0 component, the bands are equally spaced. The fundamental spectra expected are sketched in figure 3; (iv) the frequency of the central band of a symmetric spectrum need not be equal to the Rayleigh frequency of a spherical droplet of the same volume.

To estimate the size of the splitting produced consider the situation when $\epsilon_4 = 0$. Then, numerically, the maximum and minimum distortions occur when $\theta = 0$, that is, when $R = \pm 0.631\epsilon_2$. If the total distortion $\epsilon_2 Y_2^0$ has a maximum value of 0.1 (10% deviation from sphericity), $\epsilon_2 = 0.159$ and the maximum frequency shift relative to the undistorted sphere is approximately 10.7%. For this case the fractional change in frequency is therefore comparable with the fractional distortion.

3.2. Constant magnetic flux density: the frequency sum rule

The second simplification is that in which the magnetic pressure depends on the instantaneous shape of the droplet but the magnetic flux density, \mathbf{B} , does not vary as the droplet vibrates.

To assess the effect of this assumption, suppose that the surface magnetic pressure can be expressed as a three-dimensional Taylor series of form

$$p^\dagger = \sum_{ijk} c_{ijk} x^i y^j z^k, \quad (3.9)$$

with the c_{ijk} constants. Expressed in spherical polar coordinates,

$$p^\dagger(\theta, \varphi) = \sum_{u,v,n} \beta_{u,n}^\nu (r_\sigma/a)^n Y_u^v(\theta, \varphi), \quad (3.10)$$

where the $\beta_{u,n}^v$ are a set of coefficients that determine the magnetic flux density. It is now possible to calculate the $\partial p^\dagger / \partial \langle l, m | \zeta \rangle$ terms that appear in (2.59), supposing that the magnetic flux, and hence the $\beta_{u,n}^v$ terms remain unchanged during droplet oscillation. With this approximation, since $\partial r_\sigma / \partial \langle l, m | \zeta \rangle = a Y_l^m$, it follows that

$$\frac{\partial p^\dagger}{\partial \langle l, m | \zeta \rangle} = \sum_{uvn} n \beta_{u,n}^v (1 + R + \zeta)^{n-1} Y_l^m Y_u^v. \quad (3.11)$$

From (3.10) and (2.43) it is apparent that R and $\beta_{u,n}^v$ are proportional, so neglecting terms of order higher than R^2 , at equilibrium, when $\zeta = 0$, this gives

$$\frac{\partial \langle j, k | p^\dagger \rangle}{\partial \langle l, m | \zeta \rangle} = \sum_{uvn} n \beta_{u,n}^v \langle j, k | u, v | l, m \rangle. \quad (3.12)$$

This result can be substituted into (2.57). Although knowledge is still lacking about the individual $\beta_{u,n}^v$ values, a useful result may be obtained by taking the trace of the block-diagonal submatrices of $\mathcal{A} \mathcal{B}^{-1} \mathcal{C}$. That is

$$\begin{aligned} \sum_m (\mathcal{A} \mathcal{B}^{-1} \mathcal{C})_{lm, lm} &= \sum_{m=-l}^{+l} l(l-1)(l+2) + \frac{la}{\gamma} \sum_m \sum_{uvn} n \beta_{u,n}^v \langle l, m | u, v | l, m \rangle \\ &- \sum_{d>1} \sum_m \langle l, m | d, e | l, m \rangle \langle d, e | R \rangle \{3l(l-1)(l+2) - \frac{1}{2}(l^2 + 5l - 2)d(d+1)\}. \end{aligned} \quad (3.13)$$

Now from the addition theorem of spherical harmonics (of which the familiar $\sin^2 \theta + \cos^2 \theta = 1$ is a particular case)

$$\sum_m \langle l, m | u, v | l, m \rangle = \begin{cases} (2l+1) Y_0^0 & \text{for } u = v = 0, \\ 0 & \text{for } u > 0, \end{cases} \quad (3.14)$$

and hence:

$$\sum_m (\mathcal{A} \mathcal{B}^{-1} \mathcal{C})_{lm, lm} = l(l-1)(l+2)(2l+1) + l(2l+1) \frac{a}{\gamma} \sum_n n \beta_{0,n}^0 Y_0^0. \quad (3.15)$$

Consider the three translational frequencies, obtained by setting $l = 1$. As the trace of a matrix is equal to the sum of its eigenvalues, then

$$\overline{\omega_\tau^2} = \frac{1}{3} \left(\frac{\gamma}{\rho a^3} \right) \sum_m (\mathcal{A} \mathcal{B}^{-1} \mathcal{C})_{1m, 1m} = \left(\frac{1}{\rho a^2} \right) \sum_n n \beta_{0,n}^0 Y_0^0, \quad (3.16)$$

where $\overline{\omega_\tau^2}$ is the mean-square angular frequency of translational motion.

Similarly for the $l = 2$ fundamental vibrational modes,

$$\overline{\omega_{\text{fundamental}}^2} = \frac{1}{5} \left(\frac{\gamma}{\rho a^3} \right) \sum_m (\mathcal{A} \mathcal{B}^{-1} \mathcal{C})_{2m, 2m} = \frac{8\gamma}{\rho a^3} + \left(\frac{2}{\rho a^2} \right) \sum_n n \beta_{0,n}^0 Y_0^0, \quad (3.17)$$

where $\overline{\omega_{\text{fundamental}}^2}$ is the mean of the squares of the fundamental, $l = 2$, frequencies. Hence

$$\overline{\omega_{\text{fundamental}}^2} = \omega_R^2 + 2\overline{\omega_\tau^2}. \quad (3.18)$$

This result shows that the mean-square frequency of the fundamental oscillation modes is raised by the influence of the magnetic field. This result is independent of any symmetry constraints on the form of the distortion from sphericity and so provides a general expression for the analysis of observed spectra.

4. The induced magnetic field

The various approximations discussed above were adopted in order to avoid the complexity of the problems associated with the presence of the levitating magnetic field. Although useful results were obtained these problems now need to be addressed. This section of the analysis therefore seeks: (i) to find the permanent distortion, R , given information about the magnetic field before the droplet is inserted; (ii) to find out how the magnetic pressure depends upon the instantaneous shape of the droplet. This will then allow the oscillation frequencies to be determined in terms of the parameters that describe the original magnetic field.

A complete analysis of the situation necessarily requires a calculation of the induced electric currents in the droplet, such as has been done by Rony (1964). However, a considerable simplification can be made by using the previous assumption that the skin depth in which the currents are induced is very small compared to the radius of the droplet. Since there is now no magnetic flux within the bulk of the fluid, the effective intensity of magnetization, I , of the droplet can be found without recourse to any detailed description of the induced electric currents.

Suppose that, before insertion of the droplet into the levitating field, the undistorted magnetic field strength is H and that, after insertion of the droplet, the total magnetic flux density external to the droplet surface is B_0 . Then

$$B_0 = \mu_0 H + I. \quad (4.1)$$

Outside the droplet, the current density is zero, so from Maxwell's equations it follows that

$$\nabla \cdot B_0 = 0 \quad \text{and} \quad \nabla \wedge B_0 = \frac{\mu_0}{c} \frac{\partial E}{\partial t}, \quad (4.2)$$

where E is the electric field strength, and c the speed of light. For all reasonable frequencies of the levitating field alternating current, this last term is negligible, so to an excellent approximation, outside the droplet, $\nabla \wedge B_0 = 0$. Therefore B_0 can be expressed as

$$B_0 = \nabla \Phi, \quad (4.3)$$

where Φ is a magnetic potential. (Note that Φ is a scalar as distinct from the more usual magnetic vector potential, A .) Since $\nabla \cdot B_0 = 0$ it follows that $\nabla^2 \Phi = 0$ and so Φ can be expanded as the series

$$\Phi = \sum_{u>0} \left\{ \mathcal{H}_u^v \left(\frac{r}{a} \right)^u + \mathcal{J}_u^v \left(\frac{a}{r} \right)^{u+1} \right\} Y_u^v. \quad (4.4)$$

Here the \mathcal{H}_u^v terms may be identified as the coefficients which determine the (known) original field; the \mathcal{J}_u^v terms describe the (induced) intensity of magnetization and may be evaluated as follows.

If there is no magnetic flux density inside the droplet, then no magnetic flux lines can penetrate the surface of the droplet; it therefore follows that the component of B_0 normal to the surface must be zero, that is

$$\hat{n} \cdot B_0(\text{surface}) = 0 \quad (4.5)$$

or
$$\nabla \Phi \cdot \nabla \sigma = 0, \quad (4.6)$$

and hence
$$\frac{\partial \Phi}{\partial r} - \frac{1}{r^2} \hat{L} \Phi \cdot \hat{L} \sigma = 0. \quad (4.7)$$

Let ξ be the total deviation from sphericity, that is

$$\xi = R(\theta, \varphi) + \zeta(\theta, \varphi, t) \tag{4.8}$$

so that $\sigma = r - a(1 + \xi)$, then, at the surface,

$$\frac{\partial \Phi}{\partial r} = \frac{1}{a} \sum_{u>0} \left\{ u \mathcal{H}_u^v \left(\frac{r}{a}\right)^{u-1} - (u+1) \mathcal{J}_u^v \left(\frac{a}{r}\right)^{u+2} \right\} Y_u^v = - (a/r^2) \hat{\mathbf{L}} \Phi \cdot \hat{\mathbf{L}} \xi \tag{4.9}$$

which gives, correct to first order in ξ ,

$$\sum_{uv} Y_u^v \{ u(1 + [u-1]\xi) \mathcal{H}_u^v - (u+1)(1 - [u+2]\xi) \mathcal{J}_u^v \} = - \sum_{uv} (\mathcal{H}_u^v + \mathcal{J}_u^v) \hat{\mathbf{L}} Y_u^v \cdot \hat{\mathbf{L}} \xi \tag{4.10}$$

or

$$d \mathcal{H}_d^e = (d+1) \mathcal{J}_d^e - \sum_{uv} \mathcal{H}_u^v \{ \langle d, e | \hat{\mathbf{L}} Y_u^v \cdot \hat{\mathbf{L}} | \xi \rangle + u(u-1) \langle d, e | u, v | \xi \rangle \} \\ - \sum_{uv} \mathcal{J}_u^v \{ \langle d, e | \hat{\mathbf{L}} Y_u^v \cdot \hat{\mathbf{L}} | \xi \rangle + (u+1)(u+2) \langle d, e | u, v | \xi \rangle \}. \tag{4.11}$$

Therefore, when $\xi = 0$, one finds

$$\mathcal{J}_u^v = \mathcal{H}_u^v u / (u+1) \tag{4.12}$$

and so, to first order in ξ , the components of the intensity of magnetization are given by

$$\mathcal{J}_d^e = \left(\frac{d}{d+1} \right) \mathcal{H}_d^e + \frac{1}{d+1} \sum_{uv} \mathcal{H}_u^v \left\{ u(2u+1) \langle d, e | u, v | \xi \rangle + \frac{(2u+1)}{(u+1)} \langle d, e | \hat{\mathbf{L}} Y_u^v \cdot \hat{\mathbf{L}} | \xi \rangle \right\}. \tag{4.13}$$

The way in which these induced field coefficients change as the droplet oscillates is now determined. From (4.8), $\partial \xi / \langle l, m | \xi \rangle = Y_l^m$ and so, correct to first order in R , it follows from (4.13) that

$$\frac{\partial \mathcal{J}_d^e}{\partial \langle l, m | \xi \rangle} = \frac{1}{d+1} \sum_{uv} \mathcal{H}_u^v \left\{ u(2u+1) \langle d, e | u, v | l, m \rangle + \frac{(2u+1)}{(u+1)} \langle d, e | \hat{\mathbf{L}} Y_u^v \cdot \hat{\mathbf{L}} | l, m \rangle \right\} \tag{4.14}$$

or

$$\frac{\partial \mathcal{J}_d^e}{\partial \langle l, m | \xi \rangle} = \sum_{uv} \frac{(2u+1)}{(u+1)} \mathcal{H}_u^v \frac{\langle d, e | u, v | l, m \rangle}{2(d+1)} \{ d(d+1) + u(u+1) - l(l+1) \}. \tag{4.15}$$

With these results it is now possible to express the magnetic pressure as a function of the droplet shape and the original magnetic field components, \mathcal{H}_u^v . From (2.8) the magnetic pressure is given by:

$$2\mu_0 p^\dagger = (\nabla \Phi)^2 = \left(\frac{\partial \Phi}{\partial r} \right)^2 - \left(\frac{\hat{\mathbf{L}} \Phi}{r} \right)^2, \tag{4.16}$$

which, from (4.9), correct to first order in the total displacement ξ reduces to

$$2\mu_0 p^\dagger = - (\hat{\mathbf{L}} \Phi / r)_\sigma^2. \tag{4.17}$$

Equation (4.4) shows that at the surface, to first order in ξ ,

$$(\hat{\mathbf{L}} \Phi)_\sigma = \sum_{uv} (\mathcal{H}_u^v + \mathcal{J}_u^v) \hat{\mathbf{L}} Y_u^v; \tag{4.18}$$

therefore the magnetic pressure on an initially spherical droplet is

$$2\mu_0 a^2 p^\dagger = -\sum_{uv} \sum_{u'v'} \chi_v^u \chi_{u'}^{v'} (\hat{\mathbf{L}}Y_u^v) \cdot (\hat{\mathbf{L}}Y_{u'}^{v'}), \quad (4.19)$$

where for convenience χ_v^u is defined as

$$\chi_v^u = \left(\frac{2u+1}{u+1} \right) \mathcal{H}_u^v. \quad (4.20)$$

The permanent distortion of the droplet may now be found. From (2.55)

$$(\hat{\mathbf{L}}Y_u^v) \cdot (\hat{\mathbf{L}}Y_{u'}^{v'}) = \frac{1}{2} \sum_{de} Y_d^e \langle d, e | u, v | u', v' \rangle \{d(d+1) - u(u+1) - u'(u'+1)\} \quad (4.21)$$

so it follows from (2.43) and (4.19), for $d > 1$, that

$$\langle d, e | R \rangle = \left(\frac{1}{2\mu_0 \gamma a} \right) \sum_{uv} \sum_{u'v'} \chi_v^u \chi_{u'}^{v'} \langle d, e | u, v | u', v' \rangle \frac{d(d+1) - u(u+1) - u'(u'+1)}{2(d-1)(d+2)}. \quad (4.22)$$

This result allows the first-order permanent distortion to the sphere to be evaluated in terms of the original magnetic field components, \mathcal{H}_u^v .

It now remains to calculate how the magnetic pressure will alter as the droplet vibrates. This is accomplished by differentiating (4.16) with respect to the component of surface movement $\langle l, m | \zeta \rangle$ to give:

$$2\mu_0 \frac{\partial p^\dagger}{\partial \langle l, m | \zeta \rangle} = \frac{2}{r_\sigma^2} (\hat{\mathbf{L}}\Phi)^2 \frac{\partial r_\sigma}{\partial \langle l, m | \zeta \rangle} - \frac{2}{r_\sigma^2} (\hat{\mathbf{L}}\Phi)_\sigma \cdot \left\{ \sum_{de} \frac{\partial (\hat{\mathbf{L}}\Phi)_\sigma}{\partial \mathcal{J}_d^e} \frac{\partial \mathcal{J}_d^e}{\partial \langle l, m | \zeta \rangle} \right\} \quad (4.23)$$

and hence

$$\frac{\partial p^\dagger}{\partial \langle l, m | \zeta \rangle} = \frac{1}{2\mu_0 a^2} \left\{ 2Y_l^m \sum_{uv} \sum_{u'v'} \chi_v^u \chi_{u'}^{v'} (\hat{\mathbf{L}}Y_u^v) \cdot (\hat{\mathbf{L}}Y_{u'}^{v'}) - 2 \sum_{de} \sum_{u'v'} \chi_{u'}^{v'} (\hat{\mathbf{L}}Y_{u'}^{v'}) \cdot (\hat{\mathbf{L}}Y_d^e) \frac{\partial \mathcal{J}_d^e}{\partial \langle l, m | \zeta \rangle} \right\}. \quad (4.24)$$

This result, together with that derived above for the permanent distortion may now be substituted in to (2.57) to give for the block diagonal, $j = k$, elements of $\mathcal{A}\mathcal{B}^{-1}\mathcal{C}$ the expression

$$\begin{aligned} 2\mu_0 \gamma a (\mathcal{A}\mathcal{B}^{-1}\mathcal{C})_{ik, lm} &= 2\mu_0 \gamma a l(l-1)(l+2) \langle l, k | l, m \rangle \\ &- \sum_{d=2} \sum_{uv} \sum_{u'v'} \chi_v^u \chi_{u'}^{v'} \frac{\langle l, k | d, e | l, m \rangle \langle d, e | u, v | u', v' \rangle}{2(d-1)(d+2)} \\ &\quad \times (d(d+1) - u(u+1) - u'(u'+1)) \\ &\quad \times (3l(l-1)(l+2) + \frac{1}{2}(l^2 + 5l - 2)d(d+1)) \\ &+ l \sum_{d=0} \sum_{uv} \sum_{u'v'} \chi_v^u \chi_{u'}^{v'} \langle l, k | d, e | l, m \rangle \langle d, e | u, v | u', v' \rangle \\ &\quad \times (l(l+1) - d(d+1) - d'(d'+1)) \\ &- \therefore \sum_{d=0} \sum_{uv} \sum_{u'v'} \chi_v^u \chi_{u'}^{v'} \frac{\langle l, k | u', v' | d, e \rangle \langle d, e | u, v | l, m \rangle}{2(d+1)} \\ &\quad \times (l(l+1) - d(d+1) - u'(u'+1))(d(d+1) + u(u+1) - l(l+1)), \end{aligned} \quad (4.25)$$

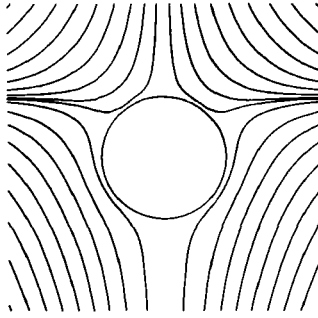


FIGURE 4. Cross-section of the spatial distribution of the magnetic flux lines of a linearly varying levitating field that contains a droplet whose centre lies one radius distance below the field centre.

which is the required result. It allows the forms of the normal modes of vibration and their associated oscillation frequencies of an actual droplet to be evaluated in terms of the parameters which describe the original undistorted levitating magnetic field.

5. Application to the case of a linear magnetic field

In this section the above general analysis is illustrated by considering the case of a droplet levitated by a particular magnetic field. A useful model to choose is that where, $\mathbf{k} \cdot \mathbf{H}$, the component of the original magnetic field strength along the z -axis, varies linearly with z . This is a good approximation to the real situation that has been found to exist along the axis of a levitating field produced by two single turn coils, aligned coaxially, but with currents flowing in opposite directions (Okress *et al.* 1952). Specifically suppose that

$$\frac{\partial(\mathbf{k} \cdot \mathbf{H})}{\partial z} = H_{zz}, \quad (5.1)$$

where H_{zz} is a constant. Then in laboratory-frame coordinates, (X, Y, Z) , since $\nabla \cdot \mathbf{H} = 0$, the magnetic field strength is given by

$$\mathbf{H} = \frac{1}{2}(-Xi - Yj + 2Zk) H_{zz}. \quad (5.2)$$

Suppose also that the centre of mass of the droplet lies a distance z_0 below the centre of the levitating field, then in the droplet frame of reference, (x, y, z) , the above linear field becomes

$$\mathbf{H} = H_z \mathbf{k} + \frac{1}{2}(-xi - yj + 2zk) H_{zz}, \quad (5.3)$$

where

$$H_z = -z_0 H_{zz}. \quad (5.4)$$

It is necessary to relate H_z and H_{zz} to the \mathcal{H}_l^m coefficients used earlier. To do this, consider the radial component of the field

$$\hat{\mathbf{r}} \cdot \mathbf{H} = (i \sin \theta \cos \varphi + j \sin \theta \sin \varphi + k \cos \theta) \cdot \mathbf{H} \quad (5.5)$$

$$= H_z \cos \theta + \frac{1}{2} r H_{zz} (3 \cos^2 \theta - 1) \quad (5.6)$$

$$= H_z \left(\frac{4}{3}\pi\right)^{\frac{1}{2}} Y_1^0 + H_{zz} \left(\frac{4}{5}\pi\right)^{\frac{1}{2}} Y_2^0 \quad (5.7)$$

$$= \hat{\mathbf{r}} \cdot \nabla \Phi = \partial \Phi / \partial r. \quad (5.8)$$

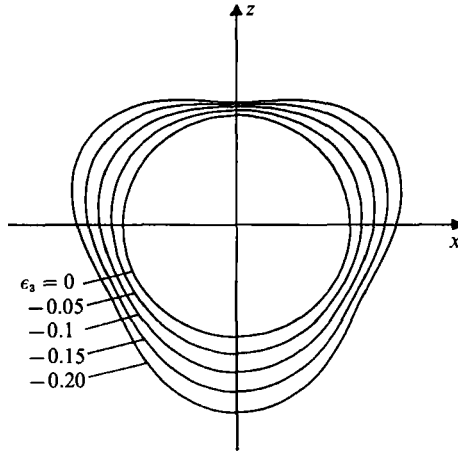


FIGURE 5. Sections in the (x, z) -plane of a permanently distorted droplet with radius at the surface given by $r_s = a(1 + \epsilon_3 Y_3^0)$ for various values of ϵ_3 . (For clarity of presentation, a has been successively increased by 10% but the actual distortion conserves the volume of the droplet.)

Comparison with (4.9) enables the only non-zero \mathcal{H}_u^v coefficients to be identified as

$$\mathcal{H}_1^0 = \left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \mu_0 a H_z \quad \text{and} \quad \mathcal{H}_2^0 = \left(\frac{1}{5}\pi\right)^{\frac{1}{2}} \mu_0 a^2 H_{zz}. \tag{5.9}$$

Note that for a spherical droplet, the only induced field coefficients will be the terms \mathcal{S}_1^0 and \mathcal{S}_2^0 . These correspond respectively to an induced magnetic dipole and a linear quadrupole, both aligned parallel to the z -axis. The spatial distribution of the magnetic flux lines of such a field is illustrated in figure 4. This has been calculated for a droplet whose centre lies one radius distance below the centre of the undistorted field.

The permanent distortion of the droplet from sphericity can now be evaluated from (4.22). There are only two non-zero magnetic field coefficients, χ_1^0 and χ_2^0 , which give three components to the deviation from sphericity. Of these, only two directly influence the $l = 2$ fundamental modes. They are

$$\langle 2, 0 | R \rangle = \left(\frac{1}{5}\pi\right)^{\frac{1}{2}} (3\mu_0 a / 8\gamma) \left\{ z_0^2 - \frac{25}{63} a^2 \right\} H_{zz}^2 \tag{5.10}$$

$$\langle 4, 0 | R \rangle = \pi^{\frac{1}{2}} (5\mu_0 a^3 / 126\gamma) H_{zz}^2. \tag{5.11}$$

It can be seen that the sign of the $\langle 2, 0 | R \rangle$ term depends on how far the droplet lies below the centre of the field. When the droplet is close to the field centre this term is negative and so imparts an oblate distortion to the droplet, but when the droplet is sufficiently deep within the field, this term becomes positive giving a prolate distortion (figure 1). The $\langle 4, 0 | R \rangle$ term is always positive and so contributes a diamond-shaped cross-section to the drop (figure 2).

The other deviation from sphericity introduced by the linear field is a coefficient of the Y_3^0 distortion:

$$\langle 3, 0 | R \rangle = \left(\frac{1}{7}\pi\right)^{\frac{1}{2}} (3\mu_0 a^2 / 10\gamma) H_z H_{zz}. \tag{5.12}$$

This term is always negative and describes an egg-shaped distortion of the droplet, the narrower part pointing downwards, as illustrated in figure 5.

The effect on the frequencies is now evaluated from (4.25). For the linear field, with

only the coefficients χ_1^0 and χ_2^0 non-zero, the block-diagonal sub-matrices of $\mathcal{A}\mathcal{B}^{-1}\mathcal{C}$ reduce to the diagonal form

$$\begin{aligned}
 2\mu_0\gamma a(\mathcal{A}\mathcal{B}^{-1}\mathcal{C})_{lm,lm} &= 2\mu_0\gamma a l(l-1)(l+2) \\
 &- \sum_{d=2}^4 \sum_{u=1}^2 (\chi_u^0)^2 \frac{\langle l, m | d, 0 | l, m \rangle \langle d, 0 | u, 0 | u, 0 \rangle}{2(d-1)(d+2)} \\
 &\quad \times (d(d+1) - 2u(u+1))(3l(l-1)(l+2) + \frac{1}{2}(l^2 + 5l - 2)d(d+1)) \\
 &+ l \sum_{d=0}^4 \sum_{u=1}^2 (\chi_u^0)^2 \langle l, m | d, 0 | l, m \rangle \langle d, 0 | u, 0 | u, 0 \rangle \\
 &\quad \times (d(d+1) - 2u(u+1)) \\
 &+ l \sum_{d=0}^4 \sum_{u=1}^2 (\chi_u^0)^2 \frac{\langle l, m | u, 0 | d, m \rangle^2}{2(d+1)} \\
 &\quad \times (d(d+1) + u(u+1)l(l+1))^2, \tag{5.13}
 \end{aligned}$$

so it is straightforward to obtain the eigenvalues. Consider first the translational modes by setting $l = 1$; this gives

$$\begin{aligned}
 (\mathcal{A}\mathcal{B}^{-1}\mathcal{C})_{1m,1m} &= \frac{(\chi_1^0)^2}{2\mu_0\gamma a} \left\{ -\frac{1}{\pi} - \langle 1, m | 2, 0 | 1, m \rangle \langle 2, 0 | 1, 0 | 1, 0 \rangle + 6 \langle 2, m | 1, 0 | 1, m \rangle^2 \right\} \\
 &+ \frac{(\chi_2^0)^2}{2\mu_0\gamma a} \left\{ -\frac{3}{\pi} + 3 \langle 1, m | 2, 0 | 1, m \rangle \langle 2, 0 | 2, 0 | 2, 0 \rangle \right. \\
 &\quad \left. + 9 \langle 1, m | 2, 0 | 1, m \rangle^2 + 32 \langle 3, m | 2, 0 | 1, m \rangle^2 \right\}. \tag{5.14}
 \end{aligned}$$

Using the values of the integrals shown in the Appendix (table 2), the frequencies are thus found to be

$$\omega_{\parallel}^2 = (3\mu_0/2\rho) H_{zz}^2 \quad \omega_{\perp}^2 = (3\mu_0/8\rho) H_{zz}^2, \tag{5.15}$$

where ω_{\parallel} is the angular frequency of the motion parallel to the z -axis, and ω_{\perp} is the angular frequency of the two degenerate translational modes in the (x, y) -plane.

For the fundamental vibrational modes, set $l = 2$ in (5.12). This gives the eigenvalues

$$\begin{aligned}
 (\mathcal{A}\mathcal{B}^{-1}\mathcal{C})_{2m,2m} &= 8 + \frac{(\chi_1^0)^2}{2\mu_0\gamma a} \left\{ -\frac{2}{\pi} - 11 \langle 2, m | 2, 0 | 2, m \rangle \langle 2, 0 | 1, 0 | 1, 0 \rangle \right. \\
 &\quad \left. + 2 \langle 1, m | 1, 0 | 2, m \rangle^2 + 16 \langle 3, m | 1, 0 | 2, m \rangle^2 \right\} \\
 &+ \frac{(\chi_2^0)^2}{2\mu_0\mu a} \left\{ -\frac{6}{\pi} + 33 \langle 2, m | 2, 0 | 2, m \rangle \langle 2, 0 | 2, 0 | 2, 0 \rangle + 12 \langle 2, m | 2, 0 | 2, m \rangle^2 \right. \\
 &\quad \left. - 16 \langle 2, m | 4, 0 | 2, m \rangle \langle 4, 0 | 2, 0 | 2, 0 \rangle + 80 \langle 4, m | 2, 0 | 2, m \rangle^2 \right\}, \tag{5.16}
 \end{aligned}$$

and hence the frequencies are

$$\left. \begin{aligned}
 \omega_{\Sigma}^2 &= \omega_{\text{R}}^2 + \overline{\omega_{\text{r}}^2} (3.832 - 0.1714(z_0/a)^2), \\
 \omega_{\text{H}}^2 &= \omega_{\text{R}}^2 + \overline{\omega_{\text{r}}^2} (3.775 + 0.5143(z_0/a)^2), \\
 \omega_{\Delta}^2 &= \omega_{\text{R}}^2 + \overline{\omega_{\text{r}}^2} (-0.9297 + 2.571(z_0/a)^2).
 \end{aligned} \right\} \tag{5.17}$$

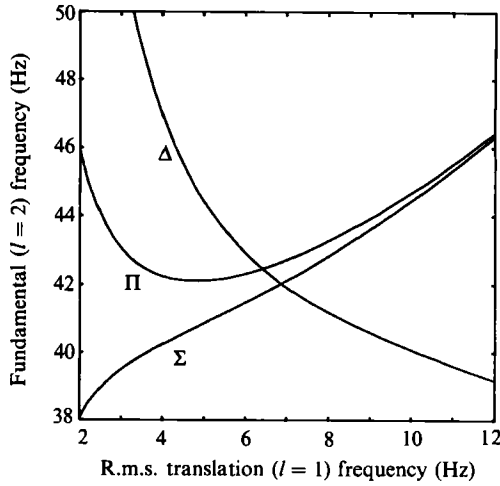


FIGURE 6. Showing how the three fundamental frequencies, ν_Σ , ν_Π , and ν_Δ of a droplet that is 4 mm in diameter and has a 40 Hz Rayleigh frequency are predicted to vary as a function of the r.m.s. translational frequency $(\bar{\nu}_T^2)^{\frac{1}{2}}$.

In the above expressions, the quantities z_0 and $\bar{\omega}_T^2$ are in fact related because of the requirement (2.44) that the lifting force must balance the weight of the droplet. From (4.19) and (4.21) and neglecting terms of order R^2 and higher, it follows that equilibrium is achieved when

$$F_z = -2\mu_0 \pi a^3 H_z H_{zz} = \frac{4}{3}\pi a^3 \rho g \tag{5.18}$$

and so from (5.14) one finds

$$z_0 = g/2\omega_T^2. \tag{5.19}$$

This result has been used to construct figure 6, where, to illustrate the effect the linear field has on the frequencies, the set of equations (5.17) has been plotted using arbitrary values of 4 mm for the droplet diameter and 40 Hz for the Rayleigh frequency. It can be seen that if the field gradient is sufficiently large (high translational frequencies), the Σ and Π frequencies converge, causing the fundamental to appear to be split into just two bands.

Some points to note are that, when the magnetic field shows a linear change of intensity in the z -direction:

(i) The translational frequencies are determined only by the density of the droplet and the magnetic field strength gradient, H_{zz} . This latter result is to be expected, since, in a magnetic field of *constant* intensity, there is no net lifting force on the droplet, and hence no restoring forces which could lead to a translational frequency.

(ii) The ω_\parallel translational frequency will always be exactly twice that of the pair of ω_\perp translations. This can provide a useful indicator of the effective linearity of the field.

(iii) The Rayleigh frequency, from which the surface tension may be evaluated, is now related to the individual vibrational frequencies from (5.17) by

$$\omega_R^2 = \frac{1}{5}(\omega_\Sigma^2 + 2\omega_\Pi^2 + 2\omega_\Delta^2) - \bar{\omega}_T^2 \left\{ 1.90 + 1.20 \left(\frac{z_0}{a} \right)^2 \right\}, \tag{5.20}$$

which may be compared to the frequency sum rule, (3.18), obtained previously. It

can be seen that the correction factor now depends upon the relative position of the droplet in the field.

6. Discussion

The calculations show that the characteristic splitting of the expected single frequency of the fundamental mode of vibration of a spherical droplet into either three, or five bands, can be explained by the deformation of the droplet from sphericity caused by the effects of the supporting electromagnetic field.

Busse (1984) has shown that an equally spaced five-band spectrum will occur in the case of a rotating spherical droplet. This work shows that a three-band spectrum will arise when the droplet is subject to a deformation which preserves an axis of rotational symmetry. If the magnetic field fails to preserve such an axis of symmetry five unequally spaced bands will occur.

Because the nature of the deformation from sphericity depends on the spatial distribution of the magnetic field, the equilibrium shape, and hence the form of the spectrum obtained, should in principle change with the relative position of the centre of mass of the droplet within the levitating field. This feature of the model is clearly amenable to experimental test.

In approaching the problem of analysing experimental spectra, the magnetic field and the induced currents which cause levitation can be regarded as a pressure acting on the surface of the droplet, in addition to the pressure already being produced by the surface tension forces. In consequence, the surface tension appears to be increased over its normal value. This requires a small correction to be made in all analyses of surface energy.

A simple model, where the magnetic flux is assumed to remain constant, has the consequence that an evaluation of the Rayleigh frequency can be made in terms of the observed fundamental oscillation frequencies, ω_i , and the frequencies of the translational motion of the droplet:

$$\omega_R^2 = \frac{1}{5} \sum_{i=1}^5 \omega_i^2 - 2\overline{\omega}_T^2. \quad (6.1)$$

The more detailed treatment of the magnetic field shows a correction factor to the Rayleigh frequency of similar magnitude for a droplet close to the centre of the field, but one that is also sensitive to the position of the droplet in the field. This position dependency could be important, if for example one were trying to measure small changes in surface tension in a series of differing physical conditions such as temperature or impurity concentration; the experimenter would need to take care that the droplet was always in the same relative position in the field.

Note also that when all five bands can be seen, the models provide a unique estimate of the Rayleigh frequency, and hence the surface tension. In the case of cylindrical symmetry, however, three bands are normally predicted. When this happens, two pairs in the five bands have identical frequencies, that is, two are doubly degenerate and (6.1) becomes

$$\omega_R^2 = \frac{1}{5}(\omega_\Sigma^2 + 2\omega_\Pi^2 + 2\omega_\Delta^2) - 2\overline{\omega}_T^2. \quad (6.2)$$

A problem of assignment now arises as any of the three bands can in principle be the non-degenerate band. It is possible only to provide upper and lower limits for the

surface energy, according to whether the non-degenerate band is either the highest, ω_{\max} or the lowest ω_{\min} of the three observed bands. Then

$$\omega_R^2 = \frac{1}{10}(3\omega_{\max}^2 + 3\omega_{\min}^2 + 4\omega_{\text{middle}}^2) - 2\bar{\omega}_r^2 \pm \frac{1}{10}(\omega_{\max}^2 - \omega_{\min}^2). \quad (6.3)$$

This problem of band assignment is the subject of a future paper.

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Appendix

The way in which the effective magnetic pressure of $B_0^2/2\mu_0$ on the surface of the droplet arises as a consequence of assuming that the magnetic flux density is excluded from the interior of the droplet can be shown as follows.

From the Maxwell equations, the current density induced by a field of the form $\mathbf{B} = H(\sigma) \mathbf{B}_0$ (equation (2.7)) is given by

$$\mu \mathbf{J} = \nabla \wedge \mathbf{B} = H(\sigma) \nabla \wedge \mathbf{B}_0 + \nabla H(\sigma) \wedge \mathbf{B}_0 \quad (A 1)$$

and since outside the drop the current density is negligible (equation (4.2) *et seq.*), $\nabla \wedge \mathbf{B}_0 = 0$, which yields

$$\mu_0 \mathbf{J} = \delta(\sigma) \nabla \sigma \wedge \mathbf{B}_0, \quad (A 2)$$

where $\delta(\sigma) = dH(\sigma)/d\sigma$ is the Dirac delta function.

From (2.6) the pressure gradient in the droplet due to the electromagnetic forces is

$$\nabla p = \mathbf{J} \wedge \mathbf{B}, \quad (A 3)$$

and since the condition $\nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{B}_0 = 0$ requires $\delta(\sigma) \mathbf{B}_0 \cdot \nabla \sigma = 0$ (cf. equation (4.5)), this gives

$$\mu_0 \nabla p = -\delta(\sigma) H(\sigma) B_0^2 \nabla \sigma. \quad (A 4)$$

The excess pressure inside the droplet is found by integrating along a line, L , which starts at a point just outside the drop and ends at a point just inside. That is

$$p^\dagger = p(\text{inside}) - p(\text{outside}) = \int_L \nabla p \cdot d\mathbf{r} \quad (A 5)$$

$$= \frac{1}{\mu_0} \int_{\sigma < 0}^{\sigma > 0} \delta(\sigma) H(\sigma) B_0^2 d\sigma \quad (A 6)$$

$$= (B_0^2/2\mu_0)_{\sigma=0}. \quad (A 7)$$

which is the required result.

	$m = 0$	$m = \pm 1$	$m = \pm 2$
$\langle 1, m 1, 0 2, m \rangle$	$1/(5\pi)^{\frac{1}{2}}$	$(3/20\pi)^{\frac{1}{2}}$	0
$\langle 1, m 2, 0 1, m \rangle$	$1/(5\pi)^{\frac{1}{2}}$	$-1/(20\pi)^{\frac{1}{2}}$	0
$\langle 1, m 2, 0 3, m \rangle$	$(27/140\pi)^{\frac{1}{2}}$	$(9/70\pi)^{\frac{1}{2}}$	0
$\langle 2, m 1, 0 3, m \rangle$	$(27/140\pi)^{\frac{1}{2}}$	$(6/35\pi)^{\frac{1}{2}}$	$(3/28\pi)^{\frac{1}{2}}$
$\langle 2, m 2, 0 2, m \rangle$	$(5/49\pi)^{\frac{1}{2}}$	$(5/196\pi)^{\frac{1}{2}}$	$-(5/49\pi)^{\frac{1}{2}}$
$\langle 2, m 2, 0 4, m \rangle$	$(9/49\pi)^{\frac{1}{2}}$	$(15/98\pi)^{\frac{1}{2}}$	$(15/196\pi)^{\frac{1}{2}}$
$\langle 2, m 4, 0 2, m \rangle$	$3/7\pi^{\frac{1}{2}}$	$2/7\pi^{\frac{1}{2}}$	$1/14\pi^{\frac{1}{2}}$

TABLE 2. Integrals required for linear field model (Albasiny, Bell & Cooper 1963)

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